Module IV

CONNECTIVITY AND PLANAR GRAPHS

* Vertex connectivity
* Edge connectivity
* Cut set and cut vertices
* Fundamental circuits
* Planar Graphs
* Kuratowski's theorem (No proof)
* Different representation of planar graphs
* Euler's theorem
* Geometric Dual.

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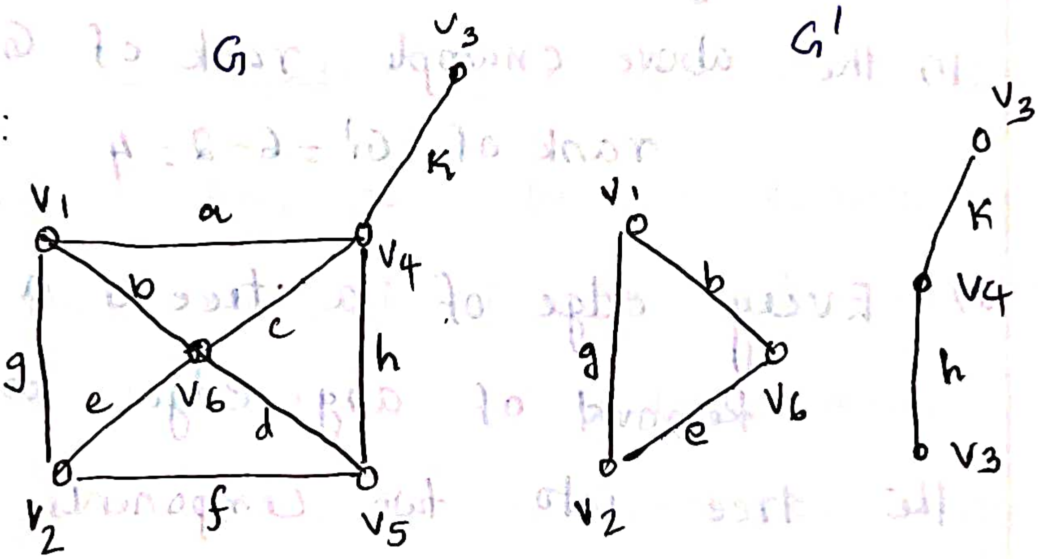
Cut-sets

In a connected graph G, a cut-set is a set of edges (minimal set) whose removal from G leaves G disconnected provided removal of no proper subset of these edges disconnects G. A cut-set always cuts a graph into two.

Eg:



$\{a, c, d, f\}$ is a cut-set which disconnects the graph into two.

There are many cut-sets in a connected graph G.

$\{a, b, g\}$, $\{a, b, e, f\}$, $\{d, h, f\}$

$\{k\}$ are some cut sets of G

But $\{a, c, h, d\}$ is not a cut set because one of the proper subset $\{a, c, h\}$ is a cut set.

Note:

1) The removal of edges in the cut set reduces the rank of the graph by one.

For a graph G, rank = $n - k$

In the above example rank of $G = 6 - 1 = 5$

rank of $G' = 6 - 2 = 4$

2) Every edge of a tree is a cut set. Removal of any edge breaks the tree into two components.

3) Properties of a cut-set

Theorem 4.1

Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.

Proof :

Let G be a connected graph & T be a spanning tree of G. Let S be an arbitrary cut set in G. We have to prove that S must contain at least one branch of T.

So assume the converse.

Assume that cut-set S & spanning tree T has no edge in common.

Then removal of the edge in the cut-set wouldn't disconnects the graph. This is a contradiction to the assumption that S is a cut-set.

Hence the theorem.

Theorem 4 : 2 :

In a connected graph G any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.

Proof

Let G be a connected graph.
Let Q be a minimal set of edges of G containing at least one branch of every spanning tree of G.

Let G-Q be the subgraph that remains after removing the edges in Q from G.

Then G-Q is clearly a disconnected graph. Because G-Q contains no spanning tree of G.

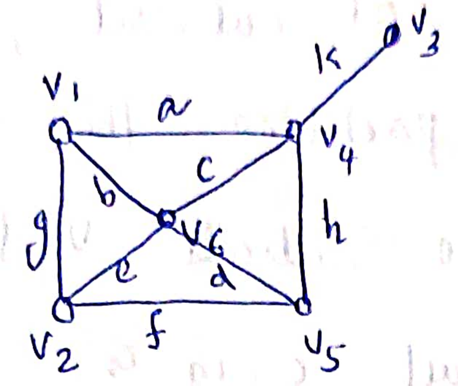
Since Q is the minimal set of edges with the property as edge e' from Q returns to G-Q will create a connected graph.

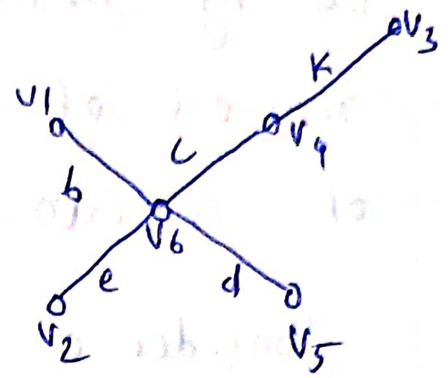
The G-Q+e will be a connected graph. ∴ Q is the minimal set of edges whose removal from G disconnects G.

Hence Q is a cut-set.

Illustrative example:

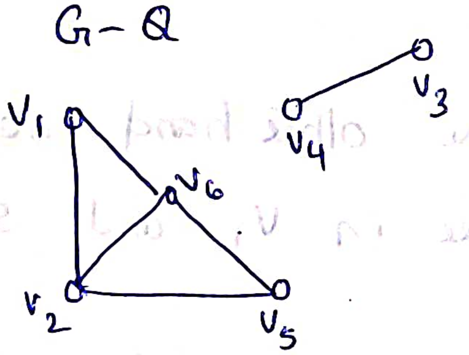
Connected graph G One spanning tree T

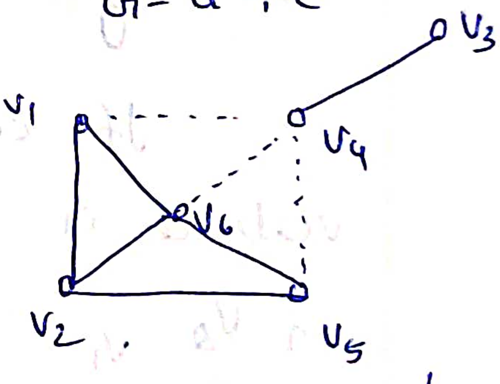




Let $Q = \{a, c, h\}$ minimal set of edges containing at least one branch (c) of spanning tree.

$G - Q$ $G - Q + e$





$G-Q$ is disconnected connected graph.

Theorem 4.3:

Every circuit has an even numbers of edges in common with any cut set.

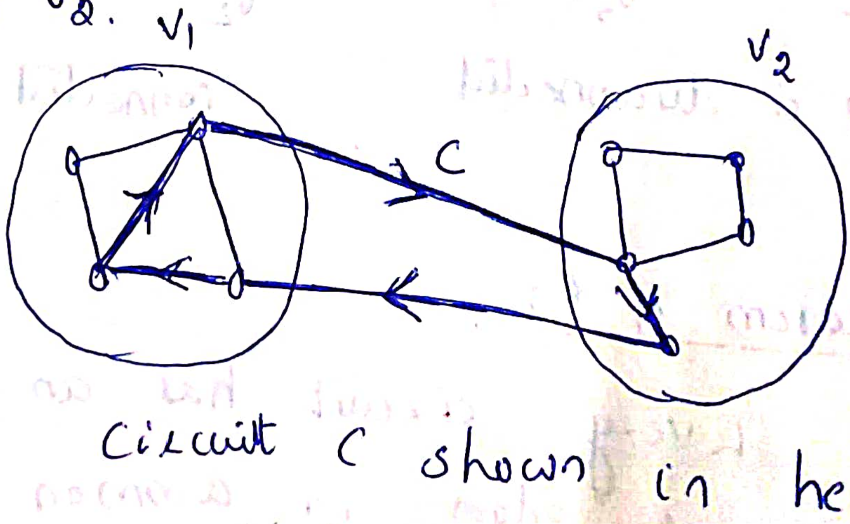
Proof

Let S be a cut-set in G. Then by definition of cut-set, the removal of S partition the vertices of G into two subsets V_1 & V_2.

Consider a circuit C in G.

If all vertices of C are entirely within set V_1 (or V_2) the number of edges common to S & C are zero (clearly even)

If on the other hand some vertices in C are in V_1 and some in V_2.



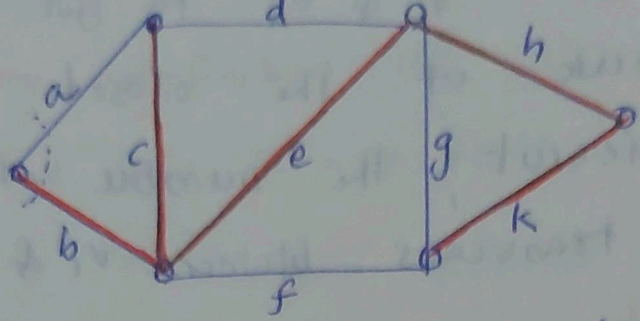
Circuit C shown in heavy lines, and is traversed along the direction of arrows we travel back & forth between two sets V_1 & V_2 to get a circuit. Because of the closed nature of a circuit, the number of edges we traverses btween V_1 & V_2 must be even.

Since S is a cut-set every edge in S has one end in V_1 and the other end in V_2, and no other edge in G has this property.

∴ The number of edges common to S & C are even.

Fundamental cut-sets.

Defn: A cut-set containing exactly one branch of a tree is called a fundamental cut-sets, w.r.t that tree. It is also called a basic cut-sets.



Fundamental cut sets are {a, b}
{a, c, d}
{d, e, f}
{h, g, f}
{f, g, k}

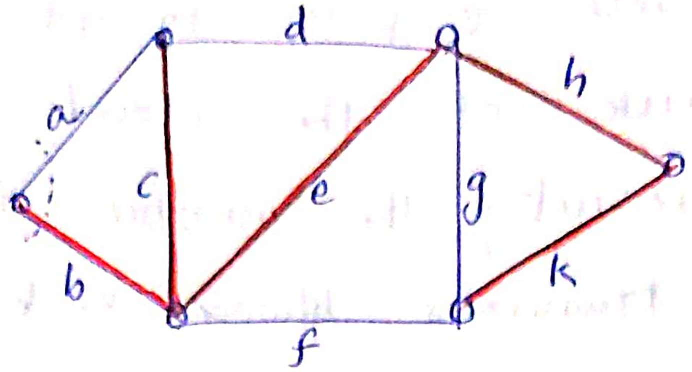
Connectivity (for connected graphs only)

Edge connectivity

The number of edges in the smallest cut-set (cut set with fewest number of edges) is defined as edge connectivity.

The edge connectivity of a tree is one.

Note: It is also defined as the minimum numbers of edges whose



Fundamental cut sets are {a, b}
{a, c, d}
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{h, g, f}
{f, g, k}

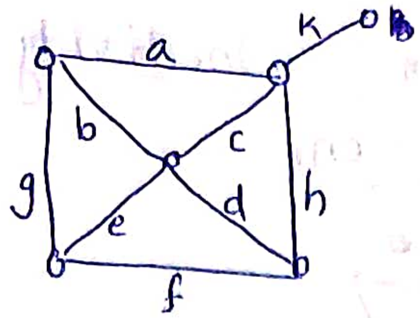
Connectivity (for connected graphs only)

Edge connectivity

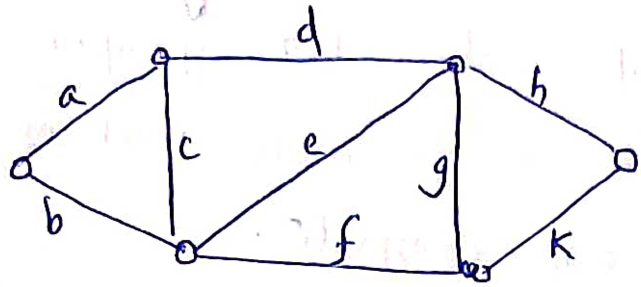
The number of edges in the smallest cut-set (cut set with fewest number of edges) is defined as edge connectivity.

The edge connectivity of a tree is one.

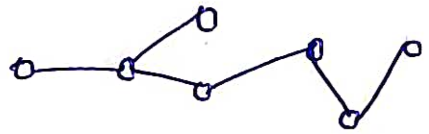
Note: It is also defined as the minimum numbers of edges whose removal disconnects the graph.

1) 

edge connectivity - 1
deletion of one edge K disconnects the graph.

2) 

Edge connectivity - 2
deletion of 2 edges disconnect graph

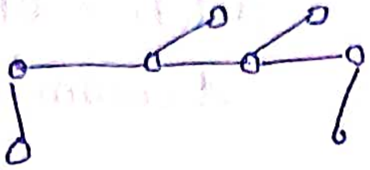
3) 

Tree - edge connectivity is one.

Vertex connectivity (Connectivity) K(G)

- The vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected.

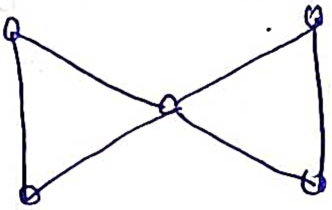
Note 1: Vertex connectivity of a tree is one.



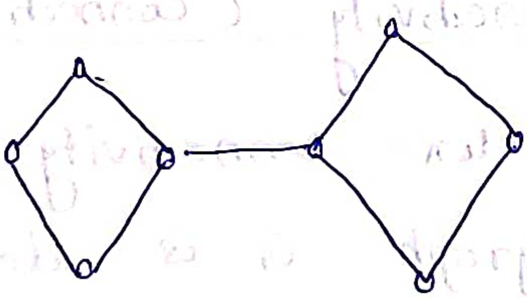
Note 2: Vertex connectivity is meaningful only for graphs that have 3 or more vertices and are not complete.

Eg:

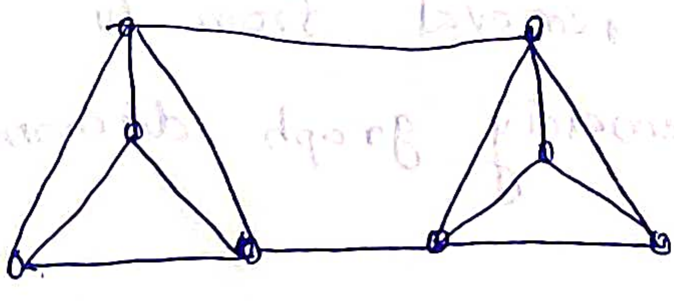
1.



2.



3.



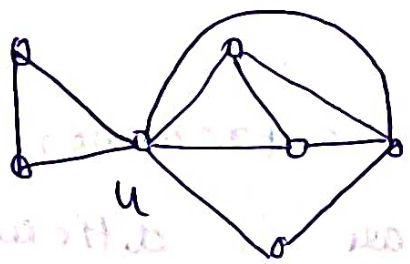
Seperable graph

A connected graph is said to be seperable if its vertex connectivity is one. All other connected graphs are called nonseperable.

Cut-vertex (Cut-node) (An articulation point)

In a seperable graph a vertex whose removal disconnects the graph is called is called a cut vertex.

Eg.



u is a cut-vertex.

Theorem

A vertex v in a connected graph is a cut vertex iff there exists two vertices x & y in G such that every path between x and y passes through v.

Proof: Necessary part:

Let v be a cut vertex of G

G−V is disconnected. Let G_1 be one of the component of G−V.

Let V_1 be the vertex set of G_1 and V_2 be the vertex set of the other component.

Let x & y be two vertices such that x is in V_1 and y is in V_2.

Consider any x-y path in G.
If the cut-vertex v does not lie in the path then this path is also in G−V.

This is a contradiction to the fact that x & y are in different components of G−V.

Hence vertex v lies in every x-y path.

Sufficient part:

Suppose v is on every x-y path. Then the vertices x & y are not connected in G−V. Thus the graph G−V is not connected. Hence by definition v is a cut-vertex.

Theorem

The edge connectivity of a graph G cannot exceed the degree of the vertex with smallest degree in G.

Proof

Let v_i be the vertex with smallest degree in G. Let $d(v_i)$ be the degree of v_i. Vertex v_i can be separated from G by removing the $d(v_i)$ edges incident on vertex v_i.

Theorem

The vertex connectivity of any graph can never exceed the edge connectivity of G.

Proof

[Removal of a vertex implies that the removal of all the edges incident on that vertex. Removal of an edge implies that the end vertices are still there]

Let α denotes the edge connectivity of G. Hence there exists a cut set S with α edges.

Let S partitions the vertices of G into subsets V_1 & V_2.

Remove at most α vertices from V_1 (or V_2) on which the edge in S are incident. This is same as the deletion of α edges in S from G.

Theorem

The maximum vertex connectivity of a graph with n vertices & e edges is $\left[\frac{2e}{n}\right]$, (the integral part of $2e/n$)

Proof

Every edge in G contributes two degrees. The total (2e) degrees is divided among n vertices. Therefore there must be at least one vertex in G whose degree is less than or equal to $2e/n$. The vertex connectivity of G cannot exceed this number by the previous theorems.

[edge connectivity cannot exceed degree of vertex with smallest degree &

vertex connectivity $\leq$ edge connectivity.

K-connected graph

A graph is said to be K-connected if the vertex connectivity of G is K.

(i), a 1-connected graph is the same as a seperable graph.

Problem

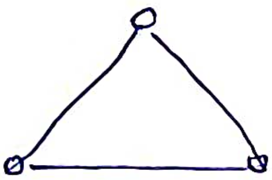
Q.1. What is the edge connectivity of the complete graph with n vertices.

Complete graph K_n is $(n-1)$ regular.

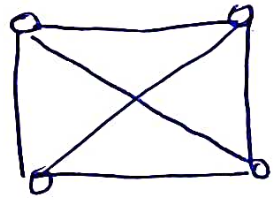
i.e., $d(v_i) = n-1$

$\therefore$ edge connectivity of complete graph $= \underline{\underline{n-1}}$

Eg: $\underline{K_3}$ $\underline{K_4}$

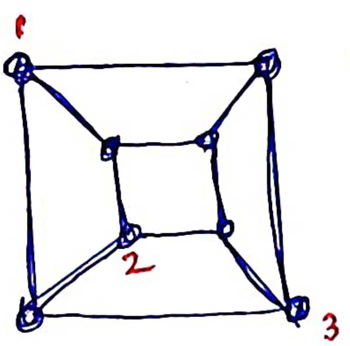


deletion of two edges disconnects the graph



deletion of 3 edges disconnects K_4

Q.2. S.T. the edge connectivity and vertex connectivity of the following graph are each equal to three.



$e = 12$

$n = 8$

Edge connectivity ≤ 3

vertex connectivity $\leq$ edge connectivity

Max. value of vertex connectivity $= \left[\frac{2e}{n}\right] = \frac{24}{8} = 3$

$\therefore$ Vertex connectivity $= 3$, edge connectivity $= 3$

Planar and Dual Graphs

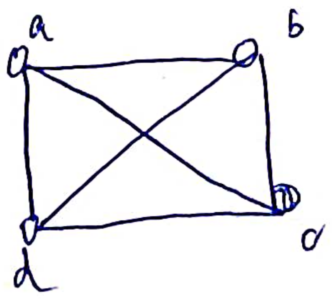
A graph is called planar if it can be drawn in the plane without any edge crossing. Such a drawing is called a planar representation of the graph.

Embedding

A drawing of a geometrical representation of a graph on any surface (2D or 3D) such that no edge intersect is called embedding.

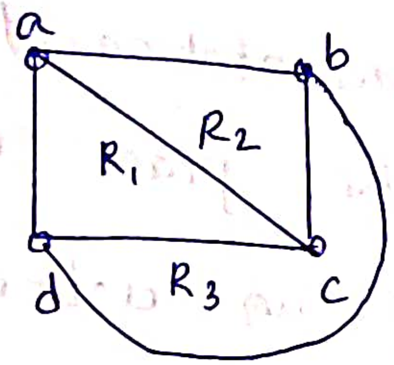
A graph that cannot be drawn on a plane without cross over between it's edges is called nonplanar.

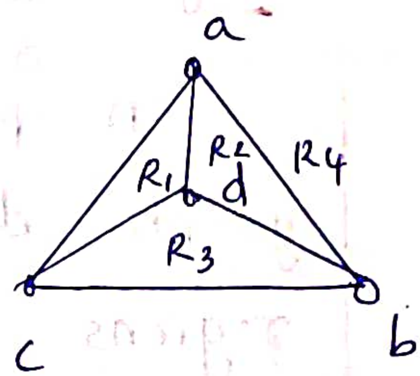
1) Is K_4 planar?



In this representation edges are crossing

Planar representation of K_4





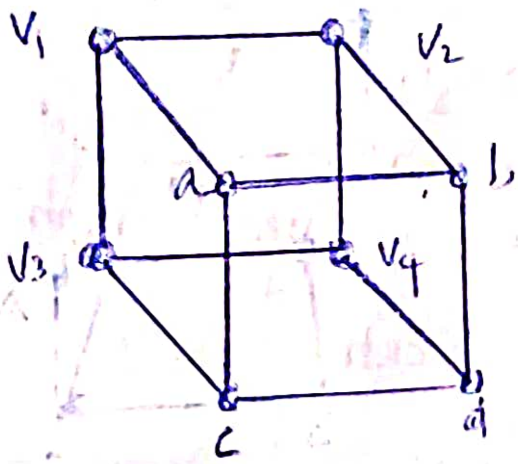
Both are same

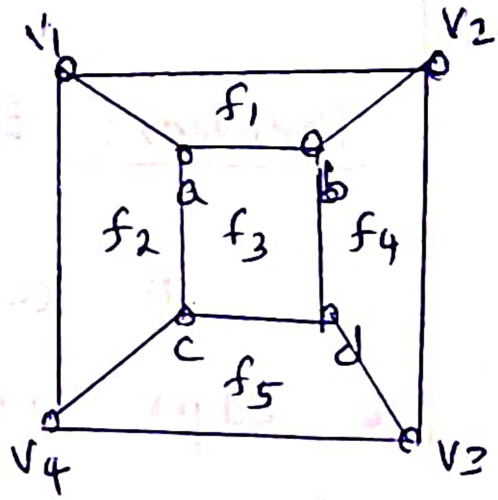
K_4 - is planar. it can be drawn without crossing.

K_4 has 4 regions.

2) Is the hypercube Q_3 planar?

(Q_3 - Hypercube)





Q_3 is planar

Regions (Windows, faces, or Meshes)

A plane representation of a graph divides the plane into regions including an unbounded region. (exterior/ outer / infinite region)

Boundary of a region.

The boundary of a region in a plane graph is the set of vertices (or the set of edges) that outline it.

Theorem: Euler's Formula

A connected planar graph with e edges and n vertices has, $f = e - n + 2$ regions.

$f = e - n + 2$ or ($r = e - v + 2$) f - faces

anothe representation e - edges

n - vertices

Proof

First we specify a planar representation of G. We will prove the theorem by constructing a sequence of subgraphs $G_1, G_2, \ldots G_e = G$, successively adding an edge at each stage.

This is done using the following inductive definition.

* Arbitrarily pick one edge of G to obtain G_1

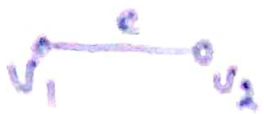
* obtain G_k from G_{k-1} by arbitrarily adding an edge that is incident with a vertex already in G_{k-1}, adding the other vertex incident with the edge if it is not already in G_{k-1}

This construction is possible because G is connected. G is obtained after 'e' edges are added.

Let, f_K - number of edges
e_K - number of edges
n_K - number of vertices of the planar graph G

Now apply induction on number of edges.

Let the number of edges $k=1$

i.e., 

then, no of region $= 1$
no. of edges $= 1$
no of vertices $= 2$

$f=1, e=1, n=2$

$f_1 = e_1 - n_1 + 2 = 1 - 2 + 2 = 1$

The planar graph has only 1 region

$\therefore$ The result is true for $k=1$

Now assume that the result is true for k number of edges

i.e., $f_k = e_k - n_k + 2$ (assumption)

Now we will prove that the result is true for $k = k+1$

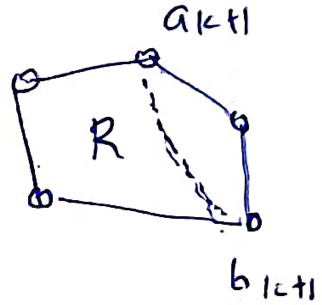
(i.e. $f_{k+1} = e_{k+1} - n_{k+1} + 2$)

Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is to added to G_k to obtain G_{k+1}. Now there are two possibilities to consider.

First case: Vertex a_{k+1} & b_{k+1} are already in G_k.

The addition of this new edge splits R into two regions (faces) consequently,

$$f_{k+1} = f_k + 1$$
$$e_{k+1} = e_k + 1$$
$$v_{k+1} = v_k$$



Now consider,

$$f_{k+1} = e_{k+1} - n_{k+1} + 2$$

i, $f_k + 1 = e_k + 1 - n_k + 2$

i, $f_k = e_k - n_k + 2$

the formula is still true.

Second case: let one of the ~~seco~~ two vertices of the new edge ~~edge~~

is not already in G_K

Suppose that a_{k+1} is in G_K and b_{k+1} is not in G_K. Adding this new edge doesnot produce any new region. because,

b_{k+1} must be in a region that has a_{k+1} on its boundary

Consequently,

$$f_{k+1} = f_k$$
$$e_{k+1} = e_{k+1}$$
$$n_{k+1} = v_k + 1$$

$$\therefore f_{k+1} = e_{k+1} - n_{k+1} + 2$$
$$f_k = e_k + 1 - (n_k + 1) + 2$$
$$f_k = e_k - n_k + 2$$

Hence the formulas is still true.

Hence $f = e - n + 2$

or

$r = e - v + 2$ //

Qn. Suppose that a connected planar graph has 30 edges. If a planar representation of the graph divides the plane into 20 regions how many vertices does the graph have?

$e = 30, \quad f = 20$

By Euler's formula,

$$f = e - n + 2$$

$$n = e - f + 2$$

$$= 30 - 20 + 2$$

$$= \underline{\underline{12}}$$

no. of vertices $= \underline{\underline{12}}$.

Qn. Suppose that a connected planar simple graph has 20 vertices each of degree 3. Into how many faces does a representation of the planar graph split the plane?

$n = 20$

$d(v) = 3$

$$2e = \sum d(v_i)$$
$$= 3 \times 20$$
$$e = \frac{60}{2} = 30$$
$$f = e - n + 2$$
$$= 30 - 20 + 2$$
$$= \underline{\underline{12}}$$

Qn). Suppose that a connected planar graph has 6 vertices. each of degree 4. Into how many regions is the plane divided by!

8

Qn) Suppose that a connected planar graph has eight vertices each of degree 3. Into how many regions is the plane divided by a planar representation of the graph?

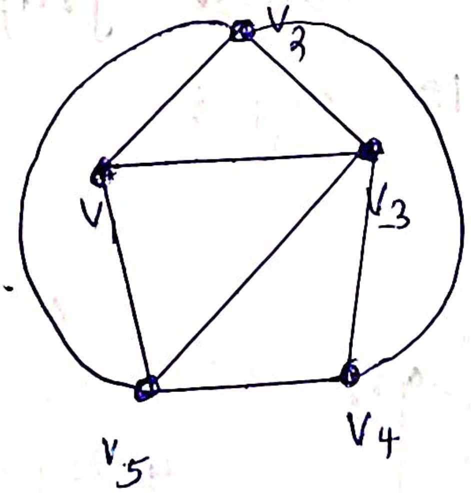
6

Different representation of planar graphs

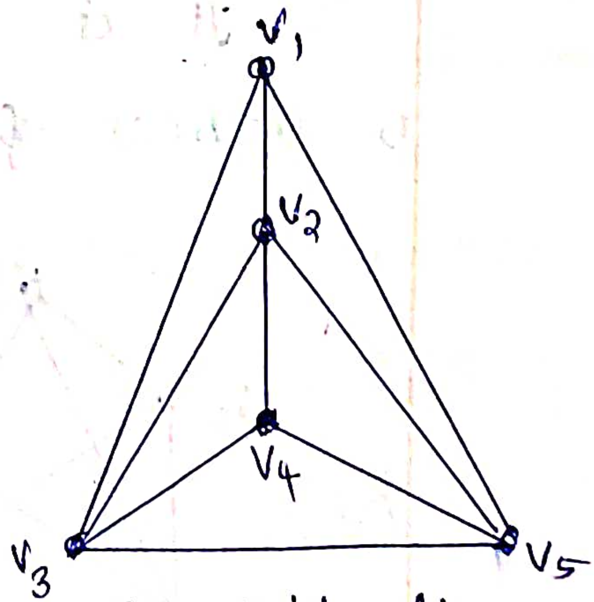
I Straight line Representation (Only for simple Planar graphs)

Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

This representation is only for simple graphs because self loops and parallel edges cannot be represented by straight lines.



$\Rightarrow$

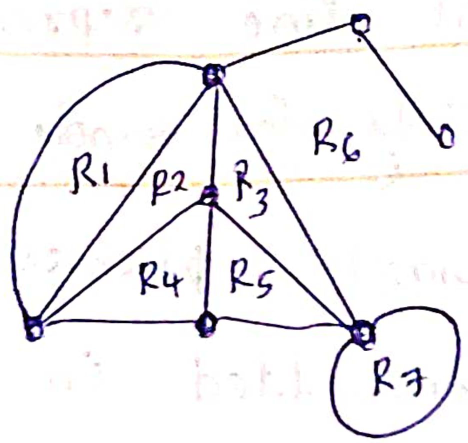


Simple planar graph $\rightarrow$ Straight line representation

II Plane Representation

(Representation with curved lines)

Eg:



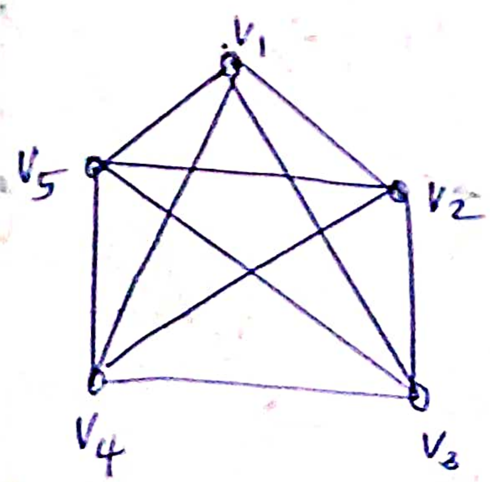
A region is characterised by set of edges (vertices)

KURATOWSKI's GRAPHS (K_5 & $K_{3,3}$)

Kuratowski's first graph K_5

It is the complete graph with 5 vertices & 10 edges

K_5 -



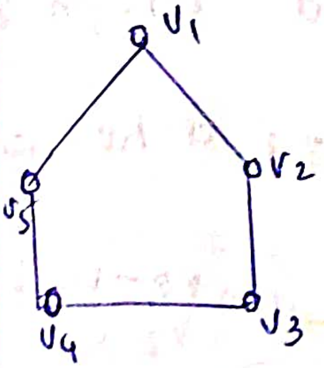
4 regular

Theorem

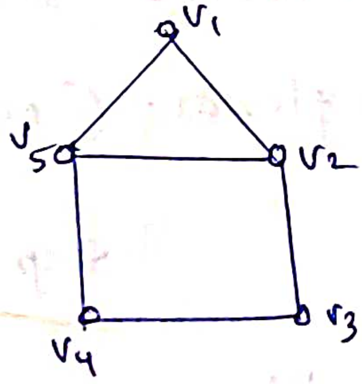
A complete graph with 5 vertices is non-planar - Kuratowski's first graph is non-planar.

K_5 is nonplanar

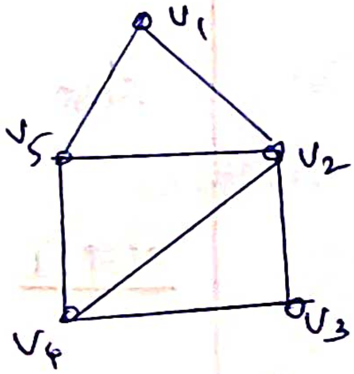
Step I



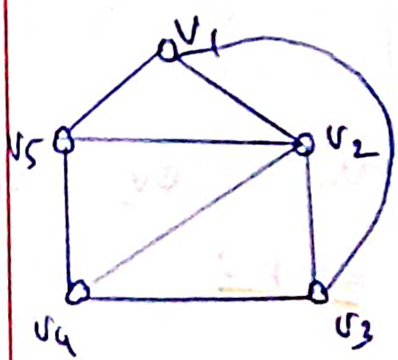
Step 2 (join v_2-v_5)



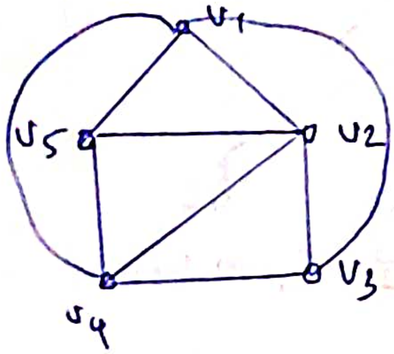
Step 3 (join v_2-v_4)



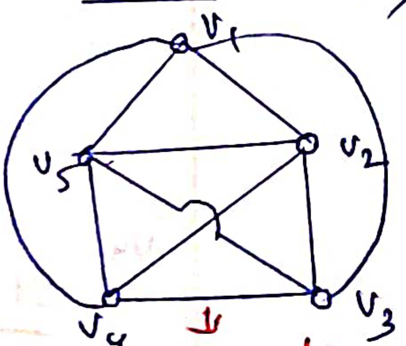
Step 4 (join v_1-v_3)



Step 5 (join v_4-v_1)



Step 6 (v_3-v_5 crosses)

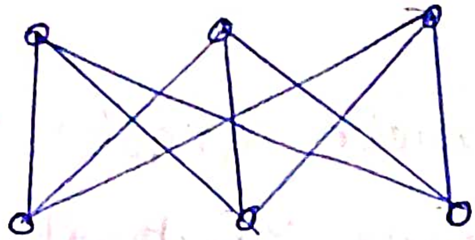


non-planar

Kuratowski's Second Graph ($K_{3,3}$)

It is a regular connected graph with 6 vertices and 9 edges.

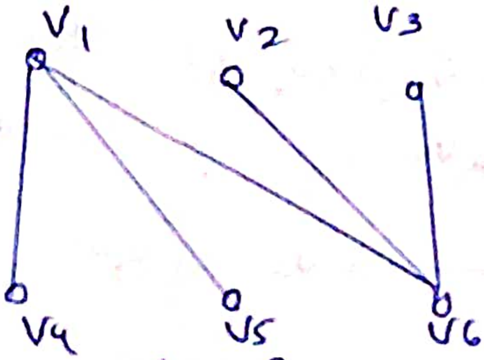
(Complete bipartite graph $K_{3,3}$)



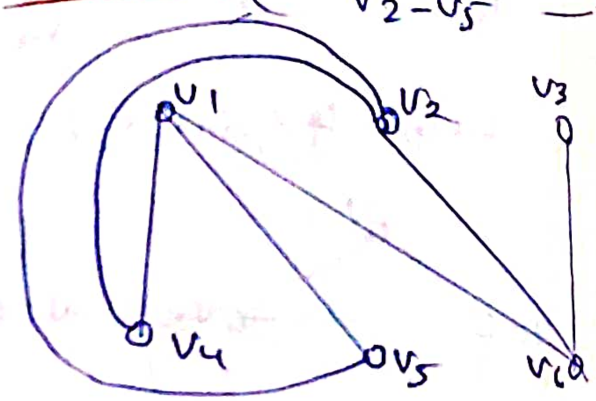
Theorem 3:

Kuratowski's second graph is also non planar. (or $K_{3,3}$ is nonplanar)

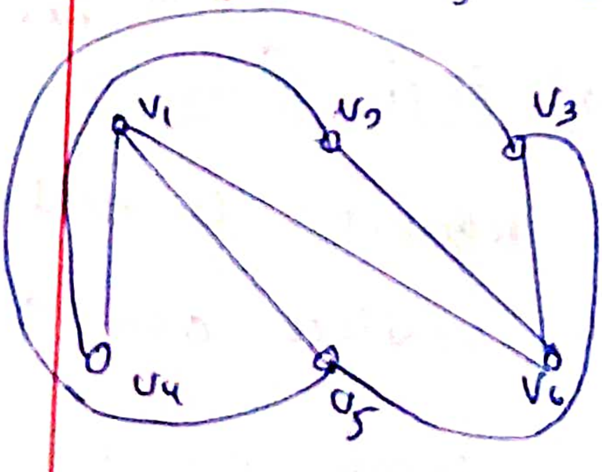
Step I



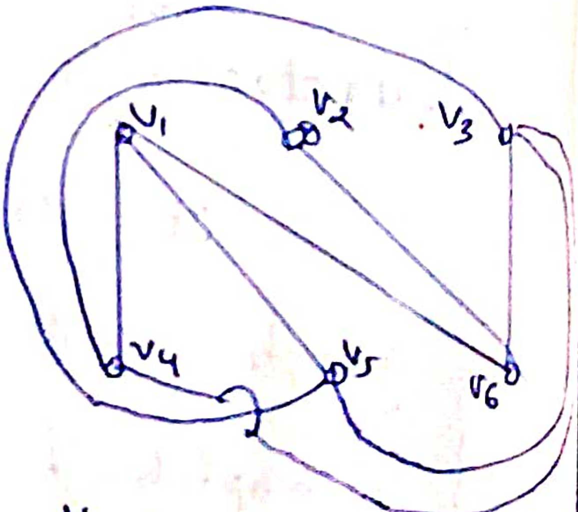
Step 2 (join $v_1 - v_4$ & $v_2 - v_5$)



step 3 (join v_3, v_5)



step 3



$K_{3,3}$ is nonplanar.

$v_3 - v_4$ cannot be drawn without crossover

Properties of Kuratowski's Graphs

1. Both are regular graphs

2. Both are non-planar

3. Removal of one edge or a vertex makes each a planar graph.

4. Kuratowski's first graph (K_5) is the non-planar graph with the smallest number of vertices.

5. Kuratowski's second graph ($K_{3,3}$) is the non-planar graph with smallest number of edges.

6. Both are the simplest non-planar graphs

7. The letter k - being for Kuratowski (K_5, $K_{3,3}$)

Corollary (i) of Euler's formula.

In any simple connected planar graphs with f regions n vertices and e edges ($e > 2$) the following inequalities must hold.

(i) $e \geq 3/2 f$

(ii) $e \leq 3n - 6$

Proof

(i) Consider a connected planar graph. Since every edge belongs to exactly two faces the ~~number~~ sum of sides of the faces in the graph is $2e$. (e - no. of edges)

Also since a region / face is bounded by at least 3 edges the minimum value of sum of sides is $3f$ (f is no. of faces)

Hence the inequality is satisfied.

i.e., $2e \geq 3f$.

i.e., $e \geq \frac{3}{2} f$

(ii) Since $e \geq \frac{3}{2} f$

substituting $f = e - n + 2$ in the above inequality,

$$e \geq \frac{3}{2}(e - n + 2)$$

$$2e \geq 3e - 3n + 6$$

i.e., $-e \geq -3n + 6$

i.e., $\underline{\underline{e \leq 3n - 6}}$

Note:

The inequality $e \leq 3n - 6$ is often used to find out whether graph is non planar.

- If a graph is planar it will satisfy the above inequality

- But a graph in which the above ~~of~~ inequalities are satisfied need not be planar.

- If the above inequalities are not satisfied by a graph then clearly the graph is non-planar.

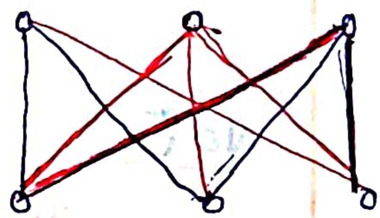
Corollary (2) of Euler's formula (for $K_{3,3}$)

If a connected planar simple graph has e edges and $n \geq 3$ vertices and no circuits of length 3 (triangle face) then

$$e \leq 2n - 4$$

Proof

There is no circuit of length 3. implies that the closed region must be bounded by at least 4 edges.



Hence if the graph is planar

$$2e \geq 4f$$

$$\text{i.e., } 2e \geq 4(e - n + 2)$$

$$2e \geq 4e - 4n + 8$$

$$-2e \geq -4n+8$$

$$e \leq 2n-4$$

Hence the proof.

---

1) P.T. Kuratowski's first graph is nonplanar using inequality.

Kuratowski's first graph is K_5

i, $e = 10$

$n = 5$

Consider

$$e \leq 3n-6$$

$$3n-6 = 15-6 = 9$$

here $e \nleq 3n-6$

Hence graph K_5 is nonplanar

2) P.T the satisfaction of inequality doesn't guarantee the planarity of a graph

Consider Kuratowski's 2nd graph

$K_{3,3}$ $\quad n=6, e=9$

$3n-6 = 3\times 6 - 9$

$= 18-9 = 9$

Here the inequality is satisfied

But $K_{3,3}$ is nonplanar.

3) Prove that Kuratowski's second graph is non-planar using Corollary 2

In $K_{3,3}$ $e = 9$
$n = 6$

$2n-4 = 12-4 = 8$

$e \nleq 2n-4$

Hence by corollary $K_{3,3}$ is nonplanar.

4) If G is a 5-regular simple graph and $|V| = 10$. P.T. G is nonplanar

$n = 10$

$d(v_i) = 5$

$\sum d(v_i) = 5\times 10 = 50$

$\therefore 2e = 50$ $e = 25$

Now consider the inequality

$e \leq 3n - 6$

here $e = 25$, $n = 10$

$3n - 6 = 3 \times 10 - 6 = 30 - 6 = 24$

$e \nleq 3n - 6$

Hence the graph does not satisfy the inequality. Hence non-planar.

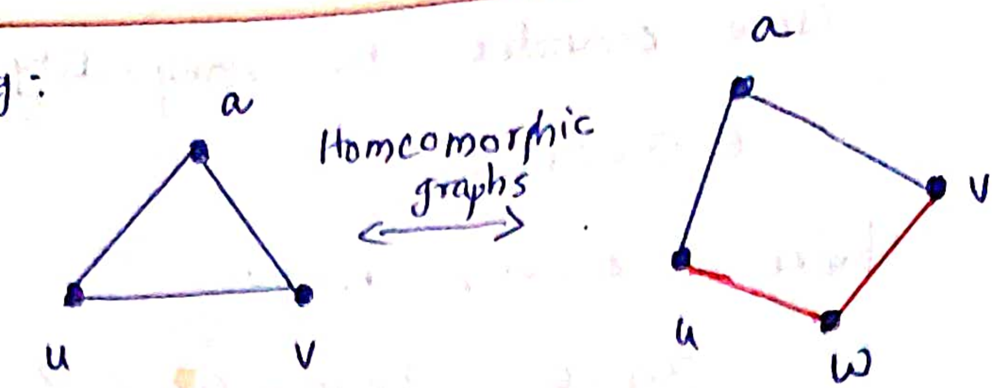
Edge subdivision or Homeomorphism

Any graph obtained by removing an edge (u, v) and adding a new vertex w together with edges (u, w) & $\{w, v\}$ from a graph G is called homeomorphic to G.

Such an operation is called elementary subdivision (insertion of vertices of degree 2).

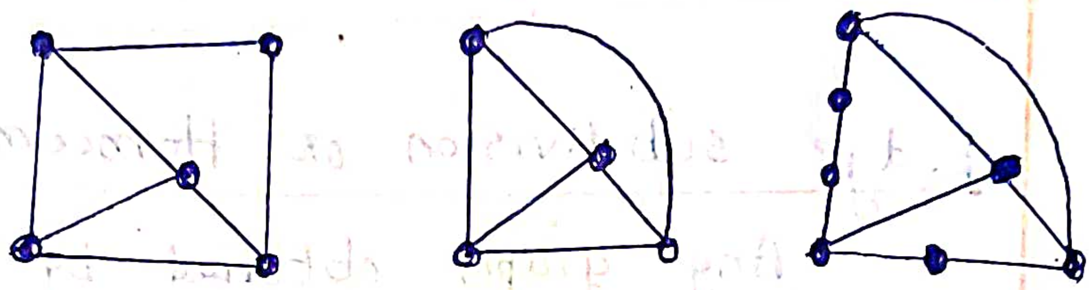
i, Two graphs are said to be homeomorphic if one graph can be obtained from the other by the creation of edges in series (i by elementary subdivision.)

Eg:



w - vertex of degree 2 is inserted b/w u & v

Eg:



All the 3 graphs are homeomorphic to each other.

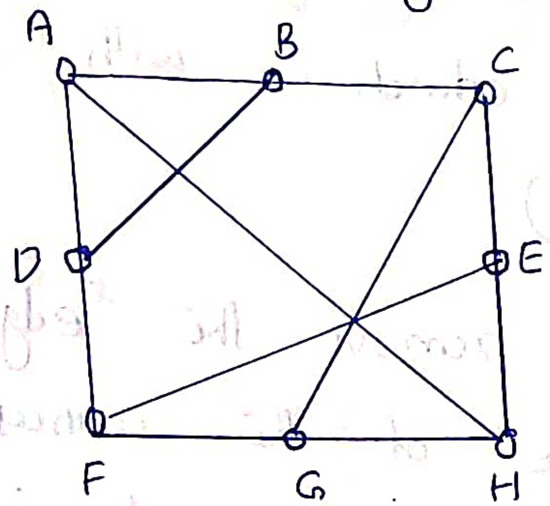
Kuratowski's theorem (Statement only)

A finite graph G is planar iff it has no subgraph that is homeomorphic to K_5 or $K_{3,3}$.

OR

A graph is nonplanar iff it has a subgraph that is homeomorphic to K_5 or $K_{3,3}$.

Q. Use Kuratowski's theorem to show that the following graph is nonplanar.



Note:

To get a subgraph we can remove edges. Also can remove vertices and hence the edges & vertices incident on these vertices.

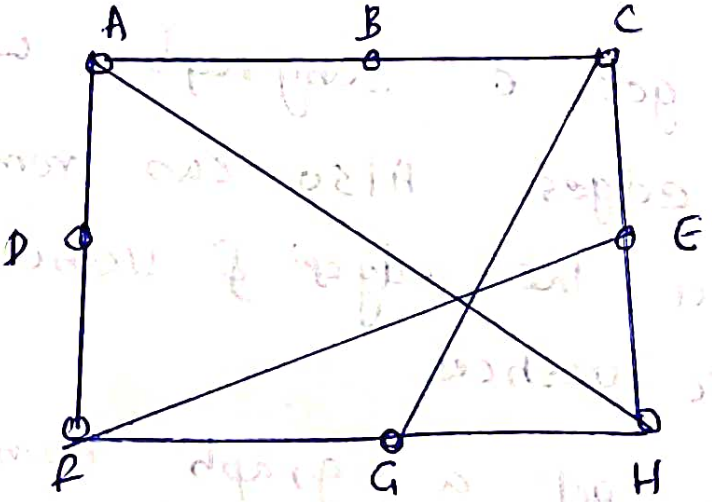
To get a graph homeomorphic we can remove vertices of degree 2.

So to check whether the graph is planar we will try to check whether there is any subgraph of G which is homeomorphic to $K_{3,3}$ or K_5

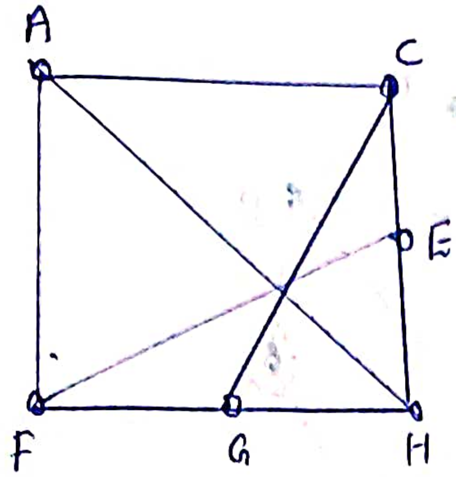
In the above graph there are 8 vertices and degree of every vertex is 3. i.e. 3 regular. ($K_{3,3}$ is 3 regular, K_5 is 4 regular & we cannot add the degree of vertices hence we try to find a graph homeomorphic to $K_{3,3}$ which is with 6 vertices & 3 regular.)

Let us remove the edge DB which makes degree of the vertices D & B, to 2. The subgraph obtained is

SUBGRAPH



Now Remove the vertices D & B of degree 2 now to get a homeomorphic graph {A,C} ,{A,F} remain as edge

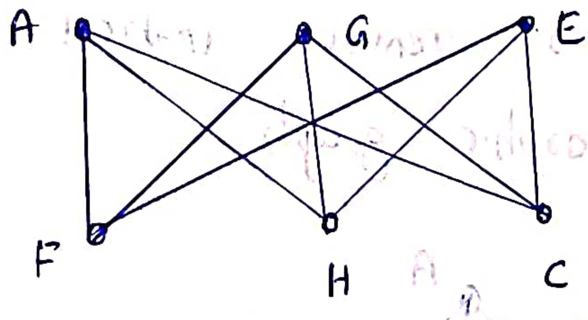


HOMEOMORPHIC GRAPH

In the graph total vertices = 6

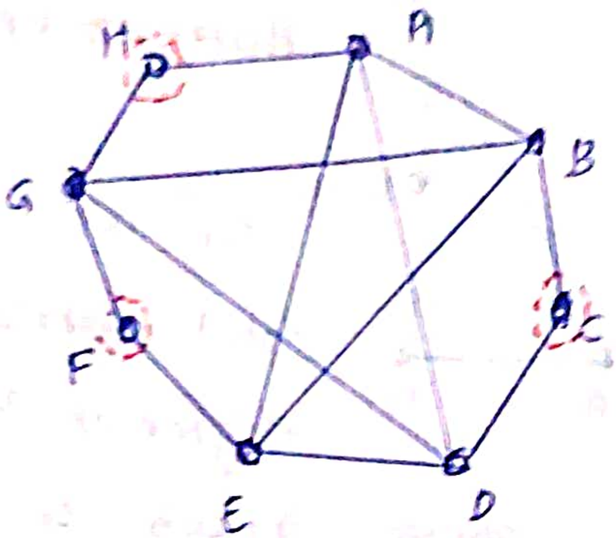
degree of vertices = 3

Now the above graph is homeomorphic to $K_{3,3}$.



Hence in the given graph there is a subgraph which is homeomorphic to $K_{3,3}$. Hence by Kuratowski's theorem the given graph is non-planar.

(2) Using Kuratowski's theorem show that the the following graph is nonplanar

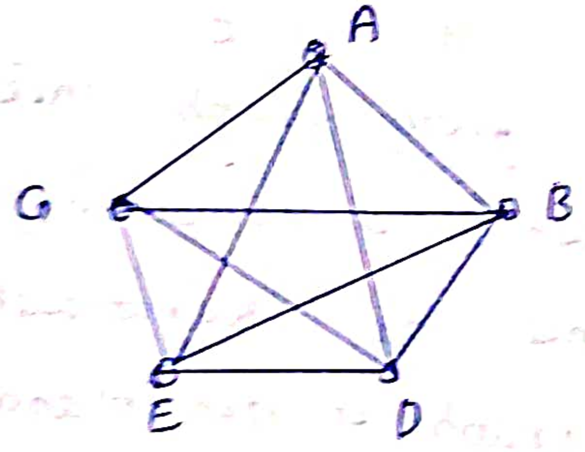


8 vertices. 5 vertices are of degree 4.

Degree of vertices C, F & H are two

So let us remove vertices to get

a homeomorphic graph



which is K_5.

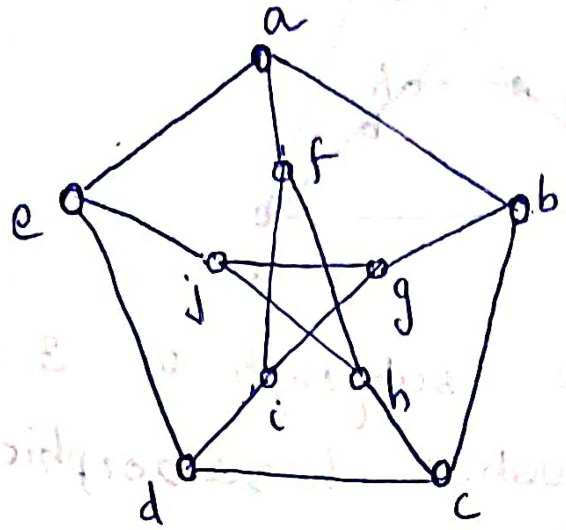
Hence the given graph has subgraph

homeomorphic to K_5

Hence by Kuratowski's theorem

~~the~~ given graph is non-planar.

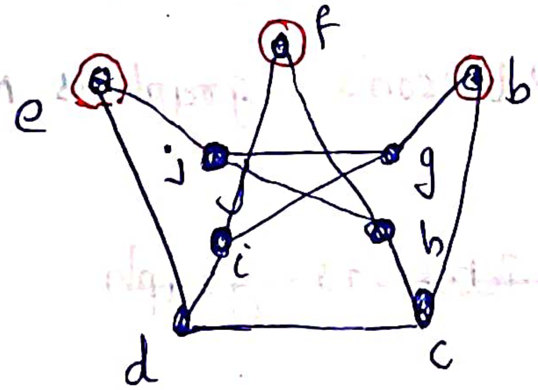
3) Use Kuratowski's theorem to show that Peterson's graph is non planar



10 vertices.
3 regular graph.

We check for subgraph homeomorphic to $K_{3,3}$.

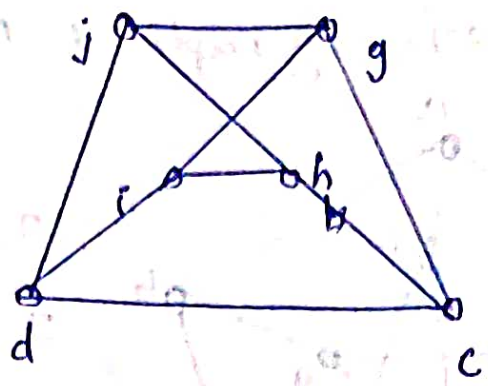
Delete vertex 'a' to get a subgraph as follows:



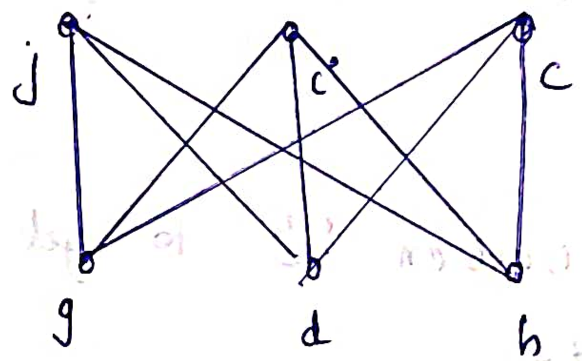
SUBGRAPH

Now vertices f, e & b are of degree two. Delete the vertices to get a homeomorphic subgraph as follows.

HOMEOMORPHIC GRAPH

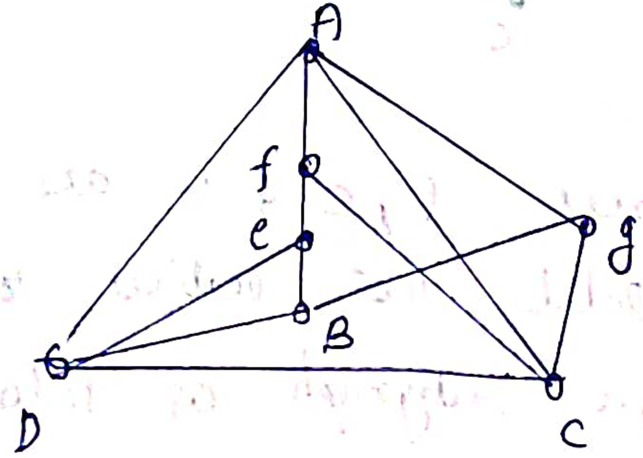


The above subgraph is 3-regular with 6 vertices homeomorphic to $K_{3,3}$.



Hence by Kuratowski's theorem the ~~give~~ Petersson's graph is non-planar.

(HW) (Qn) ~~Note: In Petersson's graph~~



Geometric Dual

Consider the plane representation of a graph G. Let $F_1, F_2, \dots F_n$ be its regions or faces. Then the geometric dual of G can be constructed by following procedure.

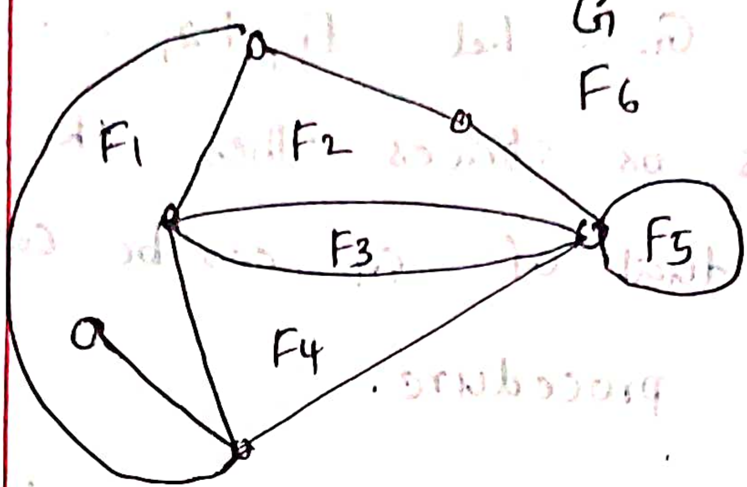
1) Place a point P_i on each face F_i

2). If two faces F_i and F_j are adjacent (have common edge) draw a line (edge) joining P_i and P_j that intersect the common edge between F_i & F_j exactly once.

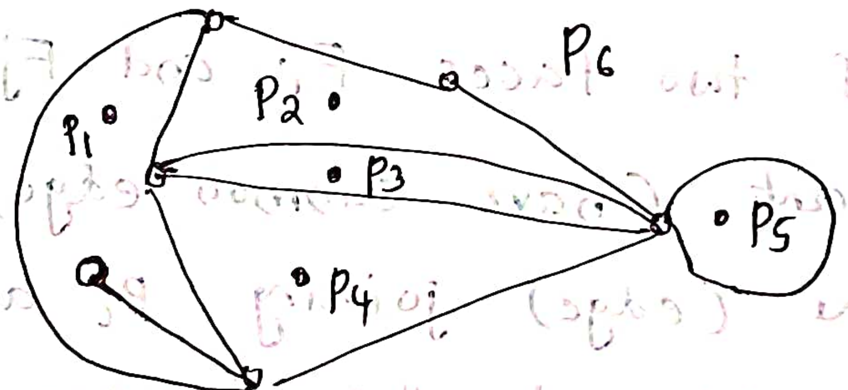
3) If there is more than one edge common between F_i & F_j draw one line between P_i & P_j for each of the common edge.

4) For an edge lying entirely in one region (ie a pendant vertex) draw a loop at the point in the region intersecting the edge exactly once.

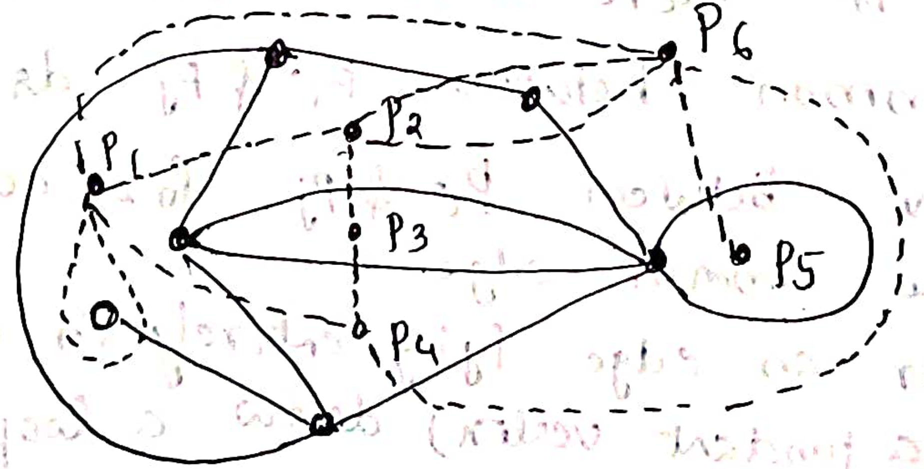
Qn. Draw the Geometric dual of the following graph.



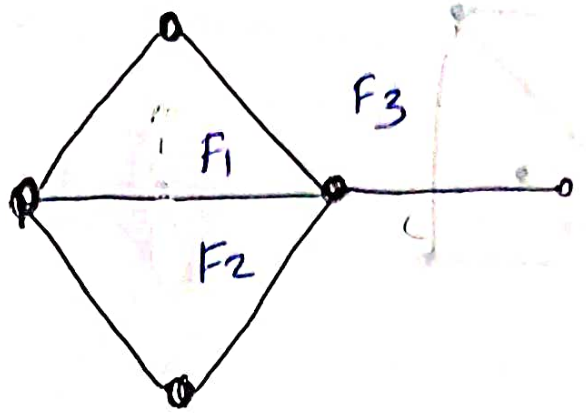
Identify faces & locate the vertices

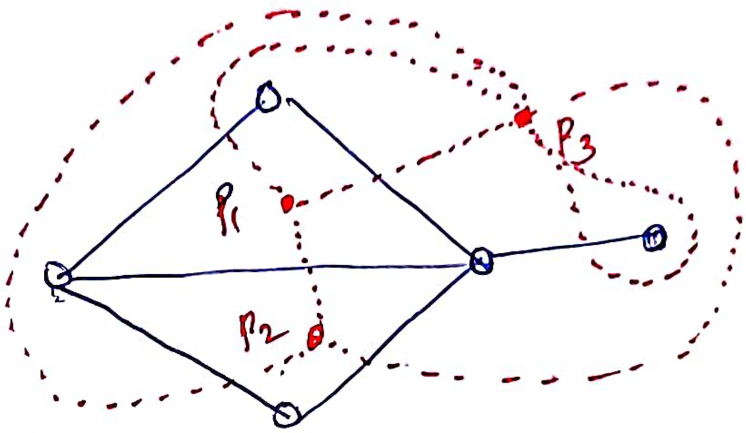


Draw edges for points s corresponding faces are adjacent

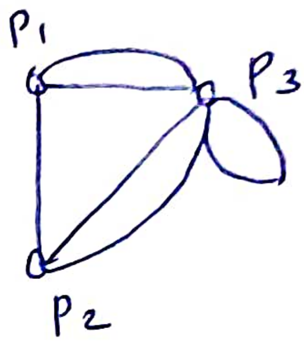


(2)

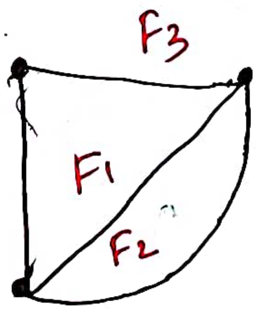


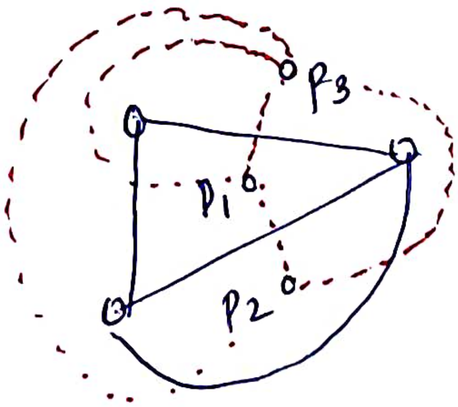


Dual G^*



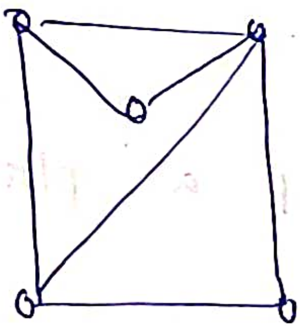
(3)



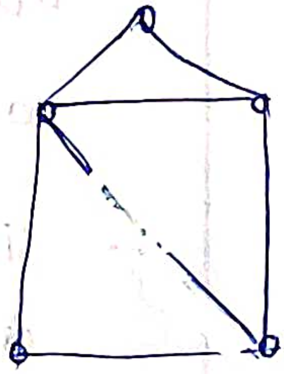


Draw the duals of following isomorphic graphs.

G_1 =



G_2 :



Results :

- Dual of isomorphic graphs are non isomorphic

- The dual of a planar graph is planar

- If n, e, f are the number of vertices, edges and faces of a connected planar graph G and n^*, e^*, f^* are those of dual G^* then,

$$n^* = f$$
$$e^* = e$$
$$f^* = n$$

- Dual of the dual of a graph is the original graph.
(for connected planar graphs)